“Hall viscosity” and intrinsic metric of incompressible fractional quantum Hall fluids.

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The (guiding-center) “Hall viscosity” is a fundamental tensor property of incompressible “Hall fluids” exhibiting the fractional quantum Hall effect; it determines the stress induced by a non-uniform electric field, and the intrinsic dipole moment on (unreconstructed) edges. It is characterized by a rational number and an intrinsic metric tensor that defines distances on an “incompressibility lengthscale”. These properties do not require rotational invariance in the 2D plane. The sign of the guiding-center Hall viscosity distinguishes particle fluids from hole fluids, and its magnitude provides a lower bound to the coefficient of the $O(q^4)$ small-q limit of the guiding center structure factor, a fundamental measure of incompressibility. This bound becomes an equality for conformally-invariant model wavefunctions such as Laughlin or Moore-Read states.

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Most treatments of the fractional quantum Hall effect (FQHE) assume rotational invariance. This has been used to demonstrate[1] that the “guiding center structure factor” $S_0(\lambda \boldsymbol{q})$ of incompressible FQHE “Hall fluids” vanishes as $O(\lambda^4)$ as $\lambda \to 0$. It is also a feature of Read's discussion[2] of their dissipationless “Hall viscosity”, which generalizes earlier results[3] by Avron *et al.* for the integer QHE. In this Letter, I show that the results [1] and [2] are not only related, but can be derived from translational invariance without invoking rotational invariance. Both properties are characterized by tensors that can be obtained from the response of FQHE wavefunctions with periodic boundary conditions (pbc's) to adiabatic changes in pbc geometry. The physical significance of the “Hall viscosity tensor” emerges: it characterizes an intrinsic electric dipole moment on unreconstructed edges of the Hall fluid, and provides an intrinsic metric deriving from the shape of the correlations that give rise to incompressibility.

A “generic” Hall system is formed by a 2D electron gas (2DEG) bound to a flat 2D quantum well embedded in a uniform 3D dielectric, through which a magnetic flux passes, with a uniform flux density $\Phi_0/2\pi\ell^2$, where $\Phi_0 = 2\pi\hbar/e$. Provided the “magnetic length” ℓ is much larger than atomic lengthscales, the clean system can be regarded as translationally-invariant in the two directions parallel to the plane of the quantum well. The one-particle eigenstates of an electron in the well have the form

$$H_0^{(1)}|\psi_{n\alpha}\rangle = \varepsilon_n|\psi_{n\alpha}\rangle, \tag{1}$$

where n is an index that combines quantum well, valley, spin, and Landau level indices. Each of these *generalized Landau levels* is a macroscopically-degenerate multiplet (with degeneracy $N_{\rm orb}$, the 2D area in units $2\pi\ell^2$) in which the “magnetic translation group” (guiding center algebra) acts:

$$\sum_\beta \langle\psi_{n\alpha}|e^{i\boldsymbol{q}\cdot\boldsymbol{R}}|\psi_{n\beta}\rangle\langle\psi_{n\beta}|e^{i\boldsymbol{q}'\cdot\boldsymbol{R}}|\psi_{n\gamma}\rangle = e^{i\frac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell^2}\langle\psi_{n\alpha}|e^{i(\boldsymbol{q}+\boldsymbol{q}')\cdot\boldsymbol{R}}|\psi_{n\gamma}\rangle. \tag{2}$$

Here $\boldsymbol{q}\times\boldsymbol{q}' \equiv \epsilon^{ab}q_a q'_b$, where $\epsilon^{ab} = \epsilon_{ab}$ is the 2D antisymmetric symbol, with an orientation (chirality) defined by the magnetic vector potential in the plane of the 2DEG: $e(\nabla_a A_b(\boldsymbol{r}) - \nabla_b A_a(\boldsymbol{r})) = \hbar\epsilon_{ab}/\ell^2$. The dynamical momentum of electrons in the 2DEG is $\pi_{ia} \equiv p_{ai} - eA_a(\boldsymbol{r}_i)$, with $[r_i^a, p_{jb}] = i\hbar\delta_{ij}\delta^a_b$, and $[r_i^a, r_j^b] = 0 = [p_{ia}, p_{jb}]$. $R_i^a = r_i^a - (\ell^2/\hbar)\epsilon^{ab}\pi_{ib}$ are the *guiding centers* that commute with the dynamical momenta, and obey the algebra $[R_i^a, R_j^b] = i\ell^2\delta_{ij}\epsilon^{ab}$. Note the use of a covariant formalism where 2D spatial coordinates r^a have upper indices $a = 1,2$, and reciprocal vectors such as wavenumbers q_a have lower indices, and only upper/lower pairs can be contracted: $\boldsymbol{q}\cdot\boldsymbol{r} \equiv q_a r^a$; this formalism makes the metric-independence of generic FQHE properties explicit.

I will assume that the low-energy effective Hamiltonian that describes degenerate perturbation theory in the residual two-body Coulomb interaction conserves the number of electrons in each level ε_n: this will be true if the splitting of the levels is large compared to the interaction strength, or if no spin-orbit coupling or interlayer tunneling mixes weakly-split levels.

The only natural metric in the problem is provided by the long-distance behavior of the non-relativistic (non-retarded) Coulomb interaction

$$V_C(\boldsymbol{r}_i - \boldsymbol{r}_j) = \frac{e^2}{4\pi\varepsilon_0\bar{\varepsilon}|\boldsymbol{r}_i - \boldsymbol{r}_j|_C}, \quad |\boldsymbol{r}|^2_C \equiv \bar{\varepsilon}\varepsilon^{-1}_{ij}r^i r^j, \tag{3}$$

where ε^{ij} is the dielectric tensor of the 3D medium surrounding the 2DEG, and $\bar{\varepsilon} = (\det|\varepsilon|)^{1/3}$; $|\boldsymbol{r} - \boldsymbol{r}'|_C$ can be called the “Coulomb distance” between two points, and Coulomb distances between points on the 2D plane

are given by $|r|_C = |r|_g \equiv (g_{ab}r^a r^b)^{1/2}$, where g_{ab} is a positive-definite 2D metric tensor with $\det|g| = 1$.

However, the long-distance part of the Coulomb interaction is not the source of incompressibility of FQHE Hall fluids, and is a complication that must be treated separately: on large lengthscales, it forces local charge-neutrality on the system, which generally requires a finite density of charged vortices (quasiparticles). As an "action at a distance", it also violates the (quasi-)local continuity equation for momentum.

FQHE incompressibility is analogous to Mott-Hubbard incompressibility, where an energy gap "U" prevents a second particle from occupying an already-occupied atomic orbital. In the $\nu = 1/3$ Laughlin state $|\Psi_L^{1/3}(g)\rangle$ [4], localized orbitals $|\psi_{nm}(\boldsymbol{r}, g)\rangle$ of level n, centered at $\boldsymbol{r}$, are eigenstates with eigenvalue $(m + \frac{1}{2})\hbar$, $m \geq 0$, of

$$L_i^z(\boldsymbol{r}, g) = \frac{\hbar}{2\ell^2} g_{ab}(R_i^a - r^a)(R_i^b - r^b), \tag{4}$$

where g_{ab} is *any* metric tensor and is a tunable parameter. In the Mott-Hubbard analogy, an energy gap prevents occupation of $|\psi_{n1}(\boldsymbol{r}, g)\rangle$ if $|\psi_{n0}(\boldsymbol{r}, g)\rangle$ is already occupied. FQHE incompressibility in general can be understood as the exclusion of additional particles from a region already occupied by a certain cluster of particles with a specific shape (controlled by the "hidden" variational parameter g in the Laughlin state).

It is convenient to separate short-range and long-range parts by writing the Coulomb interaction as the sum of a long-range part $V_C(|\boldsymbol{r}|_C^2 + a^2)^{1/2})$ and a short range part $V_0(\boldsymbol{r}, a)$ which is the interaction screened by a fictitious conducting plane parallel to the 2DEG, set back a (Coulomb) distance $a/2$ from it. The long-range part cancels the fictitious image-charge potentials. The distance a should be larger than the intrinsic "incompressibility lengthscale", but is otherwise arbitrary.

The 2D electron density is related to the 3D electron density by $\rho(\boldsymbol{r}) = \int dz\, \rho^{(3D)}(\boldsymbol{r}, z)$, and given by

$$\rho(\boldsymbol{r}) = \sum_n \int \frac{d^2q}{(2\pi)^2} e^{-i\boldsymbol{q}\cdot\boldsymbol{r}} f_n(\boldsymbol{q}) \langle \bar{\rho}_n(\boldsymbol{q}) \rangle. \tag{5}$$

Here $f_n(\boldsymbol{q})$ (where $f_n(0) = 1$) is a form factor associated with the one-particle wavefunctions of level n, and $\bar{\rho}_n(\boldsymbol{q})$ is the Fourier-transformed *guiding-center density*

$$\bar{\rho}_n(\boldsymbol{q}) = \sum_i P_i^n e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}, \tag{6}$$

where P^n is the projection into level n. If (as usually assumed) there is isotropy and Galilean invariance in the plane of the 2DEG,

$$f_n(\boldsymbol{q}) = L_n(\tfrac{1}{2}|q|_g^2\ell^2) e^{-\frac{1}{4}|q|_g^2\ell^2}, \tag{7}$$

where n is the Landau index and L_n are Laguerre polynomials. Here the metric that defines $|q|_g$ derives from the *Galilean effective mass tensor* $m^* g_{ab}$ and only coincides with the Coulomb metric if there is isotropy in the 2D plane, but if isotropy is broken (*e.g.*, by "tilting" the magnetic field), is distinct from it.

The guiding-center densities obey the Lie algebra[1] of generators of "diffeomorphisms of the quantum plane"

$$[\bar{\rho}_n(\boldsymbol{q}), \bar{\rho}_{n'}(\boldsymbol{q}')] = 2i\delta_{nn'} \sin(\tfrac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell^2)\bar{\rho}_n(\boldsymbol{q}+\boldsymbol{q}'). \tag{8}$$

In a uniform Hall fluid $\langle\bar{\rho}_n(\boldsymbol{q})\rangle = 2\pi\nu_n\delta^2(\boldsymbol{q}\ell)$, where ν_n is the filling factor of level n. The regularized densities $\bar{\rho}_n(\boldsymbol{q}) - \langle\bar{\rho}_n(\boldsymbol{q})\rangle$ also obey (8), with $\lim_{\lambda\to 0} \bar{\rho}_n(\lambda\boldsymbol{q}) = 0$. Two important subalgebras are those of the generators of translations and linear deformations of the plane: If $\bar{\rho}(\boldsymbol{q}) = \sum_n \bar{\rho}_n(\boldsymbol{q})$

$$\lim_{\lambda\to 0} \;\; \nabla_q^a \bar{\rho}(\lambda\boldsymbol{q}) \to -i\lambda\ell^2\epsilon^{ab}\hbar^{-1}P_b, \tag{9}$$

$$\lim_{\lambda\to 0} \;\; \tfrac{1}{2}\nabla_q^a\nabla_q^b \bar{\rho}(\lambda\boldsymbol{q}) \to (i\lambda\ell)^2\Lambda^{ab}. \tag{10}$$

Note that $[P_a, P_b] = \bar{\rho}(0)(\hbar^2/\ell^2)\epsilon_{ab}$, which vanishes if $\bar{\rho}(\boldsymbol{q})$ is the regularized density, so the uniform ground state of the Hall fluid consistently obeys $P_a|\Psi_0\rangle = 0$. The three independent components $\Lambda^{ab} = \Lambda^{ba}$ are generators of unitary transformations $U(\alpha) = \exp i\alpha_{ab}\Lambda^{ab}$, where $U(\alpha)R_i^a U(-\alpha) = \lambda_b^a(\alpha)R_i^b$ is a linear transformation of guiding centers that preserves their algebra; they satisfy the $SO(2,1)$ Lie algebra

$$[\Lambda^{ab}, \Lambda^{cd}] \;=\; \tfrac{1}{2}i\left(\epsilon^{ac}\Lambda^{bd} + \epsilon^{ad}\Lambda^{bc} + \epsilon^{bc}\Lambda^{ad} + \epsilon^{bd}\Lambda^{cd}\right) \tag{11}$$

with quadratic Casimir $C_2 = -\det|\Lambda|$. If there is rotational invariance with metric g,

$$L^z(g) \equiv L^z(0, g) = \hbar g_{ab}\Lambda^{ab}. \tag{12}$$

Now consider an incompressible Hall fluid subject to a slowly varying local potential $V(\boldsymbol{r})$, with the Hamiltonian

$$H = H_0 + V_1 \equiv \int d^2\boldsymbol{r}\, h_0(\boldsymbol{r}) + V(\boldsymbol{r})\bar{\rho}(\boldsymbol{r}), \tag{13}$$

where $h_0(\boldsymbol{r})$ is the quasilocal part of the Hamiltonian derived from the cyclotron motion kinetic energy and short-range part of of the Coulomb interaction, and $V(\boldsymbol{r})$ is the local potential derived from both external potentials and the long-range part of the Coulomb interaction. Here, it is assumed that $V(\boldsymbol{r})$ and $\rho(\boldsymbol{r}) \equiv \langle\bar{\rho}(\boldsymbol{r})\rangle$ are slowly varying, so all significant Fourier components have $|q|_C\ell \ll 1$. The local continuity relations for particle density $\rho(\boldsymbol{r})$ and momentum density $\pi_a(\boldsymbol{r})$ are

$$\partial_t\rho + \nabla_a j^a = 0, \quad \partial_t\pi_a + \nabla_b\sigma_a^b + \rho\nabla_a V = 0, \tag{14}$$

where $j^a(\boldsymbol{r})$ is the current density and $\sigma_b^a(\boldsymbol{r})$ is the local stress tensor. Note that local momentum conservation is violated by the "body-force" field $\nabla_a V(\boldsymbol{r})$.

The effect of $V(\boldsymbol{r})$ can be treated in linear response inside a bulk region of the fluid where $|V(\boldsymbol{r}) - V(\boldsymbol{r}')| <$

Δ, the threshold for exciting topological excitations. For $\nu = \sum_n \nu_n$,

$$j^a = \frac{\nu}{2\pi\hbar}\epsilon^{ab}\nabla_b V, \quad \sigma^a_e = \frac{1}{2\pi}\epsilon_{eb}\Gamma^{abcd}_A\nabla_c\nabla_d V. \quad (15)$$

The pressure $p(\boldsymbol{r}) = \sigma^a_a(\boldsymbol{r})$ vanishes inside the bulk regions of the Hall fluid, because the edge states shield them from forces applied to the edge: if the "container" is squeezed, the edge-current increases, as does the potential at the edge, but provided the condition $|V(\boldsymbol{r})-V(\boldsymbol{r}')| < \Delta$ remains true inside the fluid, no effects are felt in the interior. The condition $p(\boldsymbol{r}) = 0$ implies $\Gamma^{abcd}_A = \Gamma^{bacd}_A$.

The stress tensor defines the force $dF_a = \sigma^b_a\epsilon_{bc}dL^c$ between the regions of fluid on either side of a cut along a line segment $d\boldsymbol{L}$. The absence of dissipation in the ground state of the fluid in the presence of the potential $V(\boldsymbol{r})$ imposes the condition $\sigma^a_b\nabla_a j^b = 0$, or $\Gamma^{abcd}_A = -\Gamma^{cdab}_A$. Therefore Γ^{abcd}_A has the form

$$\Gamma^{abcd}_A = \pi\left(\epsilon^{ac}Q^{bd} + \epsilon^{ad}Q^{bc} + \epsilon^{bc}Q^{ad} + \epsilon^{bd}Q^{ac}\right), \quad (16)$$

where $Q^{ab} = Q^{ba}$ is *symmetric*. The expression (15) can be reformulated as a "Hall viscosity" relation by introducing the drift velocity field $v^a(\boldsymbol{r}) = \hbar^{-1}\ell^2\epsilon^{ab}\nabla_b V$: then $\sigma^a_b = \eta^{ac}_{bd}\nabla_c v^d$, where $\eta^{ac}_{bd} = (\hbar/2\pi\ell^2)\epsilon_{be}\epsilon_{df}\Gamma^{aecf}_A$.

The tensor Q^{ab} also has the physical significance of describing the *intrinsic electric dipole moment* on the (unreconstructed) boundary of a Hall fluid in its ground state, or, more generally, on the boundary between two Hall fluids with different intrinsic tensors Q^{ab}. If $d\boldsymbol{L}$ is an infinitesimal line segment on the (static) boundary, it must follow an equipotential: $\boldsymbol{\nabla}V\cdot d\boldsymbol{L} = 0$. The discontinuity in the stress tensor field leaves a net stress force on the boundary, where $dF_a = \Delta\sigma^b_a dL^b = -dp^b\nabla_b E_a$, where $eE_a = -\nabla_a V$, and

$$dp^a = e\Delta Q^{ab}\epsilon_{bc}dL^c. \quad (17)$$

For the boundary to remain static, $d\boldsymbol{p}$ must correspond precisely to an intrinsic electric dipole that experiences a compensating force from the electric field gradient to balance the stress discontinuity, and (17) may be regarded as a fundamental relation of Hall fluids. Stability requires the dipole to point in the direction of decreasing electric charge density, so $\Delta\nu\Delta Q^{ab}$ is positive.

This allows a direct calculation of ΔQ^{ab} from the edge properties. A "Landau gauge" basis of one-particle eigenstates $\boldsymbol{n}\cdot\boldsymbol{R}|\Psi_n(k)\rangle = k\ell|\Psi_n(k)\rangle$ is used to describe a translationally-invariant straight boundary along the line $\boldsymbol{n}\cdot\boldsymbol{r} = 0$. Occupation numbers $\nu_n(k)$ vary continuously between limits $\nu^{(\pm)}_n$ for $\boldsymbol{n}\cdot\boldsymbol{r}\to\pm\infty$, where k is proportional to the distance from the boundary. The total occupation number $\nu(k) = \sum_n \nu_n(k)$ has a singularity at $k = 0$ (with a form predicted by the conformal field theory of the edge[5]), because an electron can be locally added gaplessly at positions on the boundary. The dipole moment per unit length of wall is the sum of two terms: an "integer QHE" term derived completely from the form factor $f_n(\boldsymbol{q})$ and the discontinuities $\nu^{(+)}_n - \nu^{(-)}_n$, and a "fractional QHE" term that is independent of the form factor, and depends on the functions $\theta(\pm k)(\nu(k)-\nu^{(\pm)})$, which vanish in the integer QHE case. Then

$$Q^{ab} = -\frac{1}{4\pi\ell^2}\sum_n \nu_n\nabla^a_q\nabla^b_q f_n(\boldsymbol{q}) + Q^{ab}_0, \quad (18)$$

where Q^{ab}_0 is the residual guiding-center contribution obtained from the sum rule

$$\int_0^\infty \frac{dk}{2\pi}(\alpha+\beta k)(\nu(k)-\nu(\infty)) = \beta Q^{ab}_0 n_a n_b. \quad (19)$$

Q^{ab}_0 cannot change adiabatically as H_0 changes, because it is fixed by the conserved momentum parallel to the edge. Note that (19) is unchanged by rescaling $n_a \to \lambda n_a$, so is correctly metric-independent. Adiabatic continuity of the edge properties as n_a changes appears to require that the sign of $Q^{ab}_0 n_a n_b$ cannot change, which implies that Q^{ab}_0 is positive or negative definite: $\det|Q_0| > 0$.

Dynamical edge modes are described[5] by one or more conformal field theories with a net chiral anomaly $\Delta\nu$. Excitation of these modes changes the magnitude of the edge dipole moment by an amount proportional to the linear momentum (Virasoro level) of the excitation, and it will be temperature-dependent in a non-universal way that depends on the (local) edge-mode velocities. A ground-state instability (reconstruction) of the edge can also change the dipole moment.

The fractional part Q^{ab}_0 can also be derived from the expression for the guiding-center stress tensor, obtained by solving the operator relation $\hbar q_b\sigma^b_a(\boldsymbol{q}) = [\pi_a(\boldsymbol{q}), H_0]$, (using $[\pi_a(0), H_0] \equiv [P_a, H_0] = 0$), unlike previous metric-based derivations[6] which appear to assume rotational invariance. Here $\pi_a(\boldsymbol{q}) = (\hbar/\ell^2)\epsilon_{ab}[R^b]_{\boldsymbol{q}}$, where I define the generalized Fourier-transformed density

$$[f(\boldsymbol{R})]_{\boldsymbol{q}} \equiv \sum_i e^{\frac{1}{2}i\boldsymbol{q}\cdot\boldsymbol{R}_i} f(\boldsymbol{R}_i) e^{\frac{1}{2}i\boldsymbol{q}\cdot\boldsymbol{R}_i}, \quad (20)$$

where *e.g.*, $\bar\rho(\boldsymbol{q}) \equiv \sum_n \bar\rho_n(\boldsymbol{q}) = [1]_{\boldsymbol{q}}$. After some simple manipulations, using $\langle[X, H_0]\rangle_V = -\langle[X, V_1]\rangle_0 + O(V^2)$ (where $\langle\ldots\rangle_V$ is the ground state expectation value in the non-uniform system perturbed by V_1), I obtain the formal results

$$Q^{ab}_0 = \frac{1}{N_{\text{orb}}}\frac{\langle\Lambda^{ab}\rangle_0}{4\pi}, \quad \Gamma^{abcd}_{0A} = \frac{-i}{N_{\text{orb}}}\langle\tfrac{1}{2}[\Lambda^{ab}, \Lambda^{cd}]\rangle_0, \quad (21)$$

where Λ^{ab} is to be understood as the regularized quantity $[\Lambda^{ab}]_0 = \lim_{\lambda\to 0}[\Lambda^{ab}]_{\lambda\boldsymbol{q}}$.

These results are in complete agreement with the earlier results[2, 3] for the case where $L^z(g)$ is the generator of a rotational symmetry, and g^{ab} is the Galilean tensor of the cyclotron orbits: $\hbar Q^{ab}/\ell^2$ has the form[2, 3]

$\eta^{(A)} g^{ab}$,

$$Q^{ab} = \frac{g^{ab}}{4\pi} \sum_n (\nu_n s_n + \frac{\gamma_n}{2}), \quad \gamma_n \equiv N_n - \nu_n N_{\rm orb}, \quad (22)$$

where $s_n = n + \frac{1}{2}$. Here $\gamma = \sum_n \gamma_n$ is a "shift" (here defined "per flux", rather than "per particle") seen in the polynomial wavefunctions that describe rotationally-invariant Hall fluids in disk or sphere geometries, and is a purely guiding-center quantity that vanishes for a completely-filled Landau level, and is odd under particle-hole conjugation. If $\nu = p/q$, then $q\gamma$ is an integer. Without regularization, the guiding center contribution $g_{ab}\Lambda^{ab}$ to $L^z/\hbar$ for a circular finite-size Hall droplet is $\frac{1}{2}NN_{\rm orb}$, but regularization removes the superextensive part $\frac{1}{2}\nu N_{\rm orb}^2$, leaving the (extensive) regularized part $\frac{1}{2}\gamma N_{\rm orb}$. For the $\nu = k/(km+2)$ Laughlin[4] ($k = 1$), Moore-Read[7] ($k = 2$), or Read-Rezayi[8] ($k > 2$) states in a single Landau level, $\gamma = \nu(m+1)$, (m is odd for fermions, even for bosons).

If a single Hall fluid is present, the condition det $|Q_0| > 0$ implies $Q_0^{ab} = \pm\frac{1}{2}|\gamma|g^{ab}/4\pi$, where $|\gamma|$ is positive rational, g^{ab} is an intrinsic metric *defined* by the incompressibility, and the sign of Q_0^{ab} distinguishes electron Hall fluids ($Q_0^{ab} > 0$) from hole Hall fluids ($Q_0^{ab} < 0$). If there are multiple (independently deformable) Hall fluids, Q_0^{ab} will be the sum of tensors associated with each fluid.

I now make the connection to the long-wavelength behavior of the guiding-center structure factor, given in terms of the total (regularized) guiding-center density operators by

$$\langle \bar{\rho}(\boldsymbol{q})\bar{\rho}(\boldsymbol{q}')\rangle_0 = 2\pi S_0(\boldsymbol{q})\delta^2(\boldsymbol{q}\ell + \boldsymbol{q}'\ell), \quad (23)$$

or in the unregularized form,

$$S_0(\boldsymbol{q}) = \frac{1}{N_{\rm orb}} \langle \bar{\rho}(\boldsymbol{q})\bar{\rho}(-\boldsymbol{q})\rangle_0 - \langle \bar{\rho}(\boldsymbol{q})\rangle_0 \langle \bar{\rho}(-\boldsymbol{q})\rangle_0. \quad (24)$$

Note that this is normalized per flux rather than per particle, and is even under a particle-hole transformation of the partially-filled Landau levels (and vanishes identically if all Landau levels are either filled or empty). $S_0(\lambda\boldsymbol{q})$ vanishes as $\lambda \to 0$ (because the total particle number is fixed), and is even in λ. Since the ground state is assumed to be incompressible, with a gap for excitations, $S_0(\lambda\boldsymbol{q})$ will be analytic in λ, and the formal expansion coefficients depend on the fluctuations $C_{mn} = \langle [(\boldsymbol{q}\cdot\boldsymbol{R})^m]_0 [(\boldsymbol{q}\cdot\boldsymbol{R})^n]_0\rangle_0 - \langle [(\boldsymbol{q}\cdot\boldsymbol{R})^m]_0\rangle_0 \langle [(\boldsymbol{q}\cdot\boldsymbol{R})^n]_0\rangle_0$. The crucial point is that $[R^a]_0 = \ell^2\epsilon^{ba}P_b/\hbar$ which is conserved, so all coefficients C_{1m} must vanish. This argument asserts incompressibility and translational invariance of the uniform Hall fluid, with *no* requirement of rotational invariance. The leading behavior of $S_0(\lambda\boldsymbol{q})$ at small λ is derives from C_{22}: $S_0(\lambda\boldsymbol{q}) = \frac{1}{4}\lambda^4\Gamma_{0S}^{abcd} q_a q_b q_c q_d \ell^4 + O(\lambda^6)$, where Γ_{0S}^{abcd} is the symmetric fourth-rank tensor

$$\Gamma_{0S}^{abcd} = \frac{1}{N_{\rm orb}} \left(\langle \tfrac{1}{2}\{\Lambda^{ab}, \Lambda^{cd}\}\rangle_0 - \langle \Lambda^{ab}\rangle_0 \langle \Lambda^{cd}\rangle_0 \right). \quad (25)$$

Evidently this can be combined with Γ_{0A}^{abcd} to form the complex Hermitian tensor $\Gamma_0^{abcd} = \Gamma_{0S}^{abcd} + i\Gamma_{0A}^{abcd}$, *i.e.*:

$$\Gamma_0^{abcd} = \frac{1}{N_{\rm orb}} \left(\langle \Lambda^{ab}\Lambda^{cd}\rangle_0 - \langle \Lambda^{ab}\rangle_0 \langle \Lambda^{cd}\rangle_0 \right). \quad (26)$$

The linear deformations that preserve the guiding-center algebra can be written $U(\boldsymbol{x}) = exp\left(i\alpha_{ab}(\boldsymbol{x})\Lambda^{ab}\right)$, where $\boldsymbol{x} = \{x^\mu, \mu = 1, 2, 3\}$ is a three-dimensional manifold corresponding to the three distinct generators Λ^{ab}. Let $|\Psi(\boldsymbol{x})\rangle = U(\boldsymbol{x})|\Psi_0\rangle$ be a deformation of the incompressible fluid ground state. The covariant derivative in the deformation parameter space $\boldsymbol{x}$ is given by

$$|D_\mu\Psi\rangle = (\mathbb{1} - |\Psi\rangle\langle\Psi|) \left| \frac{\partial \Psi}{\partial x^\mu} \right\rangle, \quad \langle \Psi | D_\mu \Psi\rangle = 0. \quad (27)$$

Then $\langle D_\mu\Psi|D_\nu\Psi\rangle = \Gamma_0^{abcd}\partial_\mu\alpha_{ab}\partial_\nu\alpha_{cd}$ is a 3×3 positive Hermitian matrix which can be obtained by studying the adiabatic deformation of a finite-size Hall fluid system with periodic boundary conditions as pbc geometry is varied; from this, Γ_0^{abcd} can be determined[9]. This generalizes the two-parameter deformation space studied in Refs. [2, 3] where rotational invariance was assumed.

If the fluid ground state is an eigenstate of $L^z(g)$, then $g_{ab}\Gamma_0^{abcd}$ must vanish, which forces Γ_{0S}^{abcd} to have the structure $\frac{1}{2}\kappa(g^{ac}g^{bd} + g^{ad}g^{bc} - g^{ab}g^{cd})$, where g^{ab} is the metric also defined by Q_0^{ab}. Γ^{abcd} can viewed as a positive Hermitian matrix $\Gamma^{\{ab\},\{cd\}}$, which leads to the bound $\kappa \geq |\gamma|$ (because $\Gamma_S^{\{ab\},\{cd\}} \pm i\Gamma_A^{\{ab\},\{cd\}}$ is positive). The value of κ is known[1] for the Laughlin states and *satisfies the bound as an equality* $\kappa = |\gamma|$. Numerical diagonalization studies[9] show this is also true for the finite-N Moore-Read states[7], suggesting it is a general property of maximally-chiral model wavefunctions derived from conformal field theory. Corrections to these wavefunctions when realistic interactions are used appear[9] to generically increase κ above the bound, but it is not yet clear whether or not this is a finite-size effect.

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